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Extreme exchangeable random order processes by positive definite functions on semigroups

1. Motivation

Often in practice jobs are partially ordered in such a way that the second job cannot begin before the first one is finished. Such situations are considered by Kyle Siegrist in his recent paper [S]. So we have a partially ordered network of jobs and are interested in, among others, the subset of jobs which are already completed at time t . Clearly this increases with t , and the mentioned subset can be viewed as partially ordered again, the order being inherited by the original order. So we have an increasing family of partially ordered sets. Such a situation asks itself for more algebraic structurisation. In fact there is the following new application of the [HR]-method.

2. The set-up of the Bauer simplex of continuous exchangeable probability measures on the set of order processes

We use here the notation and terminology of [HR2000]. In particular \mathcal{V} is the set of all reflexive, transitive (*but not necessarily anti-symmetric*) relations — called by us **partial orders** — on M , where M is a countable infinite set, and we recall that \mathcal{V} carries a natural metrisable topology and that the diagonal relation D is the neutral element of the semigroup (\mathcal{V}, \vee) .

Definition 1. We denote by \mathcal{Y} the set of all left-continuous increasing maps $Y : [0, \infty[\rightarrow \mathcal{V}$ with $Y(0) = D$.

Definition 2. For $Y, Z \in \mathcal{Y}$ the join $Y \vee Z \in \mathcal{Y}$ is given by $(Y \vee Z)(t) := Y(t) \vee Z(t)$.

Proposition 3. (\mathcal{Y}, \vee) is an idempotent abelian semigroup with neutral element the constant D and absorbing element the map which switches from D to $M \times M$ immediately at $t = 0$.

Definition 4. By the **support** of a $Y \in \mathcal{Y}$ we understand the set $\langle Y \rangle := \bigcup_{t \in [0, \infty[} \langle Y(t) \rangle \subseteq M$.

Definition 5. $\mathcal{Z} := \{Y \in \mathcal{Y} \mid \langle Y \rangle \text{ is finite}\}$

Lemma 6. $\langle D \rangle = \emptyset$ and $\langle Y \vee Z \rangle \subseteq \langle Y \rangle \cup \langle Z \rangle$ for all $Y, Z \in \mathcal{Y}$, so \mathcal{Z} is a sub-semigroup of \mathcal{Y} .

Definition 7. For $Y, Z \in \mathcal{Y}$ we say $Y \leq Z$ if and only if $Y \vee Z = Z$.

Lemma 8. This is the case if and only if $Y(t) \leq Z(t)$ for all $t \in [0, \infty[$.

Lemma 9. “ \leq ” is a partial order on \mathcal{Y} .

Definition 10. For $Z \in \mathcal{Z}$ we put $Q_Z := \{Y \in \mathcal{Y} \mid Z \leq Y\}$.

Lemma 11. For all $Y, Z \in \mathcal{Z}$ we have $Q_{Y \vee Z} = Q_Y \cap Q_Z$.

Definition 12. A sub-semigroup I of \mathcal{Z} is called **left-hereditary** if for all $Z_1, Z_2 \in \mathcal{Z}$ $Z_1 \leq Z_2 \in I$ implies $Z_1 \in I$.

Proposition 13. The join $\bigvee_{Z \in I} Z$ is left-continuous again.

Proof. For this let us remember why the semigroup operation $\vee : \mathcal{V} \times \mathcal{V}$ is continuous. We see \mathcal{V} as a closed subset of the compact metrisable space $\{0, 1\}^{M \times M}$. There is a natural mapping $p : \{0, 1\}^{M \times M} \rightarrow \mathcal{V}$ which assigns to any $\tilde{V} \in \{0, 1\}^{M \times M}$ the smallest partial order containing \tilde{V} . It is easily seen that p is continuous, on the basis of the explicit construction with the chains and because the subsets Q_U (with $U \in \mathcal{U}$) generate the topology of \mathcal{V} .

The semigroup operation \vee on \mathcal{V} is now the obvious composition of continuous maps

$$\mathcal{V} \times \mathcal{V} \mapsto \{0, 1\}^{M \times M} \times \{0, 1\}^{M \times M} \mapsto \{0, 1\}^{M \times M} \mapsto \mathcal{V},$$

so $\vee : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$ is continuous. The problem with infinitely many factors instead of just two factors is the map in the middle in the above chain composition. I do not see that it is continuous. However, for our left-continuity we only need continuity from below. So, in terms of subsets of $M \times M$, the question is, if $A_{mn} \nearrow A_m$ as $n \rightarrow \infty$ for each m , whether or not $\bigcup_m A_{mn}$ converges to $\bigcup_m A_m$. It is definitively increasing in n , so the question is whether or not

$$\bigcup_n \bigcup_m A_{mn} = \bigcup_m \bigcup_n A_{mn}$$

is true. Trivially, it is true, completing the proof of the proposition.

Proposition 14. *If I is a left-hereditary sub-semigroup of \mathcal{Z} and $Z \in \mathcal{Z}$ is such that $Z \leq \bigvee_{Y \in I} Y$, then for each $\varepsilon > 0$ $Z(t)$ is \leq the maximum of finitely many $Y \in I$, taken at the time $t + \varepsilon$, for all t .*

Proof. For all $j, k \in M$ we put $t_0 = t_0(j, k) := \inf\{t \in [0, \infty[| (j, k) \in Z(t)\}$ with the usual convention $\inf(\emptyset) = \infty$. For each pair (j, k) with $t_0(j, k) < \infty$ and each $\varepsilon > 0$ there is a $\tilde{Y}_{jk} \in I$ such that we have $(j, k) \in \tilde{Y}_{jk}(t_0 + \varepsilon)$. We define $Y_{jk}(t)$ to be D on $[0, t_0 + \varepsilon]$ and to be $\tilde{Y}_{jk}(t_0 + \varepsilon)$ on $]t_0 + \varepsilon, \infty[$, and because of left-hereditarity we have still $Y_{jk} \in I$. For $j, k \in \langle Z \rangle$ with $t_0(j, k) = \infty$ we put Y_{jk} to be the constant D , which is always $\in I$. So we have $Z(t) \leq \bigvee_{j, k \in \langle Z \rangle} Y_{jk}(t + \varepsilon)$ with $\bigvee_{j, k \in \langle Z \rangle} Y_{jk} \in I$, and the proposition is proved.

Next, in order to endow \mathcal{Y} with a suitable topology, so to introduce in a useful way the space $M_+^1(\mathcal{Y})$ of (Radon) probability measures on \mathcal{Y} , we need to resort to a **Skorokhod**-like topology on \mathcal{Y} . For this we recall that \mathcal{V} can be seen as a subset of $\{0, 1\}^{M \times M}$, and we start with the basic set $\{0, 1\}$ in the place of \mathcal{V} , denoting by \mathcal{Y}' the set of all left-continuous increasing maps $Y' : [0, \infty[\mapsto \{0, 1\}$ with $Y'(0) = 0$. Any such map can be identified with its jump point $\in [0, \infty]$, and we let \mathcal{Y}' inherit its (compact) topology from there. Our multidimensional set \mathcal{Y} consists of certain maps from $[0, \infty[$ to $\mathcal{V} \subset \{0, 1\}^{M \times M}$, and viewed as maps from $[0, \infty[$ to $\{0, 1\}^{M \times M}$ this splits into the component maps from $[0, \infty[$ to $\{0, 1\}$, which are all left-continuous and increasing. In this way

Definition 15.

\mathcal{Y} inherits its topology from the isomorphic set

$$\tilde{\mathcal{Y}} := \{f \in [0, \infty]^{(M \times M) \setminus D} \mid f(j, l) \leq \max(f(j, k), f(k, l)) \text{ for all pairwise different } j, k, l \in M\},$$

$[0, \infty]^{(M \times M) \setminus D}$ being endowed with the product topology, which is compact by Tychonov's theorem and metrisable, and \mathcal{Y} being compact as a closed subset of $[0, \infty]^{(M \times M) \setminus D}$.

Definition 16. *By $M_+^1(\mathcal{Y})$ we denote the space of all Borel probability measures on \mathcal{Y} , all of which are automatically Radon, \mathcal{Y} being endowed with the above compact metrisable topology.*

Proposition 17. *The Borel σ -algebra on \mathcal{Y} is generated by the closed subsets Q_Z .*

Proof. In the framework of the isomorphic description of \mathcal{Y} given by Definition 15, denoting the canonical projections from $\tilde{\mathcal{Y}}$ to $[0, \infty]$ by pr_{jk} , Q_Z corresponds to the intersection of finitely many sets of the form $pr_{jk}^{-1}([0, t])$, where the t 's are the switching times of Z (since we require the switching times of the Y 's in Q_Z to be \leq those of Z), so the sets Q_Z are closed.

The Borel σ -algebra on $[0, \infty]^{(M \times M) \setminus D}$ is generated by the closed subsets, and the “finite-dimensional” closed subsets (“ $\times [0, \infty] \times [0, \infty] \times \dots$ ”) already suffice to generate this, since any other closed subset is the intersection of countably many “finite-dimensional” ones, the finite-dimensional projections being continuous. As we know from $[0, \infty]^n$, the σ -algebra generated by the finite-dimensional closed intervals $[\vec{0}, \vec{t}]$ is equal to the one generated by all closed sets, and the intervals $[\vec{0}, \vec{t}]$ correspond to our sets Q_Z . Finally, the transition from $[0, \infty]^{(M \times M) \setminus D}$ to \mathcal{Y} is achieved by taking the trace σ -algebra on the closed subset $\tilde{\mathcal{Y}}$, concluding the proof of the Proposition.

Theorem 18. *A function $\varphi : \mathcal{Z} \mapsto \mathbb{R}$ is positive definite with respect to \vee , normalised (i.e. $\varphi(D) = 1$) and continuous from below if and only if*

$$\varphi(Z) = \mu(Q_Z) \text{ for all } Z \in \mathcal{Z}$$

for a (uniquely determined) $\mu \in M_+^1(\mathcal{Y})$.

Proof. For any $\mu \in M_+^1(\mathcal{Y})$ the function $\varphi : \mathcal{Z} \mapsto [0, 1]$ defined by $\varphi(Z) = \mu(Q_Z)$ is positive definite and fulfills $\varphi(D) = 1$, as is easily seen in the same way as in previous work of Paul Ressel and myself. Moreover it is continuous from below, because this is totally analogous to the continuity *from above* for distribution functions of probability measures on (certain Borel-measurable subsets of) $[0, \infty]^n$, $Q_Z = \{Y \in \mathcal{Y} | Y \geq Z\}$ being the event that the switching times of Y are all \leq those of Z , where only finitely many switching times of Z are allowed to be $< \infty$.

Conversely, if φ is any positive definite function with $\varphi(D) = 1$, there is a unique Radon probability measure ν on $\hat{\mathcal{Z}}$ representing φ via

$$\varphi(Z) = \int 1_I(Z) d\nu(I) = \nu(\{I \in \mathcal{I} | Z \in I\}) ,$$

where \mathcal{I} denotes the set of all left-hereditary sub-semigroups of \mathcal{Z} , and where the identification of $I \in \mathcal{I}$ with $1_I \in \{0, 1\}^{\mathcal{Z}}$ is used to topologise \mathcal{I} . We define $h : \mathcal{I} \mapsto \mathcal{Y}$ by

$$h(I) := \bigvee_{Z \in I} Z .$$

In the following argument we mean by the sub-index “ $-\varepsilon$ ” the shift operation $f \mapsto f(\bullet - \varepsilon)$ (where for times $t \leq 0$ $f(t) := D$) and by the sub-index “ $+\varepsilon$ ” we mean $f \mapsto f(\bullet + \varepsilon)$. Note that in \mathcal{Z} $Z^{(1)} = Z^{(2)}$ is equivalent to $Z_{-\varepsilon}^{(1)} = Z_{-\varepsilon}^{(2)}$, but not equivalent to $Z_{+\varepsilon}^{(1)} = Z_{+\varepsilon}^{(2)}$. So in the argument below $Z_{-\varepsilon} \in I$ clearly implies $Z = (Z_{-\varepsilon})_{+\varepsilon} \in I_{+\varepsilon}$, and $Z \in I_{+\varepsilon}$ implies $Z_{-\varepsilon} \in (I_{+\varepsilon})_{-\varepsilon}$. Here, for I , first taking “ $+\varepsilon$ ” and afterwards “ $-\varepsilon$ ” means replacing the elements of I on $[0, \varepsilon]$ by D , and the result of this is still in I because of the left-hereditarity of I , so we luckily arrive at $Z_{-\varepsilon} \in I$ as desired. With the first of the following “ \supseteq ” being by Proposition 14, we have

$$\{I \in \mathcal{I} | Z_{-\varepsilon} \in I\} = \{I \in \mathcal{I} | Z \in I_{+\varepsilon}\} \supseteq \{I \in \mathcal{I} | Z \leq h(I)\} = h^{-1}(Q_Z) \supseteq \{I \in \mathcal{I} | Z \in I\} ,$$

where the ν -measure of the leftmost resp. rightmost side is $\varphi(Z_{-\varepsilon})$ resp. $\varphi(Z)$. For $\varepsilon \searrow 0$ the switching times of $Z_{-\varepsilon}$ converge from above to those of Z , which means $Z_{-\varepsilon} \nearrow Z$ in \mathcal{Z} . So with the additional assumption that φ is continuous from below, $\varphi(Z_{-\varepsilon})$ converges to $\varphi(Z)$. This shows that h is measurable with respect to the ν -completion of the σ -algebra on \mathcal{I} and that $\varphi(Z) = \nu(h^{-1}(Q_Z)) =: \mu(Q_Z)$ with μ being the image measure $\nu^h \in M_+^1(\mathcal{Y})$ as desired, finishing the proof of the theorem.

Corollary 19. In $M_+^1(\mathcal{Y})$ a sequence $(\mu_n)_{n \in \mathbb{N}}$ converges to μ if and only if the corresponding positive definite functions fulfill $\varphi_n(Z) \rightarrow \varphi(Z)$ for all $Z \in \mathcal{Z}$ with $\mu(\partial Q_Z) = 0$.

Proof. By the portmanteau theorem, the implication “ \Rightarrow ” is obvious. Conversely, if $\varphi_n(Z) \rightarrow \varphi(Z)$ for all $Z \in \mathcal{Z}$ with $\mu(\partial Q_Z) = 0$, then by the compactness of $M_+^1(\mathcal{Y})$ we have along some sub-sequence $\mu_n \rightarrow \tilde{\mu}$ and $\varphi_n(Z) \rightarrow \tilde{\varphi}(Z)$ for all Z with $\tilde{\mu}(\partial Q_Z) = 0$, so $\varphi(Z) = \tilde{\varphi}(Z)$ for all Z with $\mu(\partial Q_Z) = \tilde{\mu}(\partial Q_Z) = 0$. By an analogous argument as with distribution functions on \mathbb{R}^n , this suffices to imply $\mu = \tilde{\mu}$ and finally $\mu_n \rightarrow \mu$ for the whole sequence.

Next we introduce our notion of **exchangeability** for measures in $M_+^1(\mathcal{Y})$. The *permutations* (finite or infinite ones, this does not matter here) of the countably infinite base set M act in the natural way on the set \mathcal{V} of partial orders, cf. [HR2000] . And on \mathcal{Y} such a permutation acts timepointwise in the same way as it acts on \mathcal{V} , and so the action carries in the canonical way over to $M_+^1(\mathcal{Y})$.

Definition 20. If a $\mu \in M_+^1(\mathcal{Y})$ is invariant under all these permutations, then we call it **exchangeable** and denote this in signs by $\mu \in M_+^{1,e}(\mathcal{Y})$.

Next we express this property by means of a mapping

$$g : \mathcal{Z} \mapsto T ,$$

where $(T, +)$ is another abelian (but not idempotent) semigroup. T is defined as the set of all isomorphism classes of \mathcal{Z} , where $Z_1, Z_2 \in \mathcal{Z}$ are called isomorphic if they are permutations of each other. $g : \mathcal{Z} \mapsto T$ is then the mapping which assigns to every $Z \in \mathcal{Z}$ its isomorphy class $g(Z) \in T$. The semigroup operation “ $+$ ” on T is defined by

$$g(Z_1) + g(Z_2) := g(Z_1 \vee Z_2) \text{ for representants } Z_1 \text{ and } Z_2 \text{ with disjoint supports.}$$

Clearly this is well-defined. In the same way as in our previous papers it is straightforward that g is strongly almost additive and hence **strongly positivity forcing**. The mapping g now expresses exchangeability as follows: let

$\mu \in M_+^1(\mathcal{Y})$; then we have the corresponding positive definite function $\varphi : \mathcal{Z} \mapsto \mathbb{R}$ with $\varphi(Z) = \mu(Q_Z)$, and μ is exchangeable if and only if $\varphi(Z)$ depends only on the isomorphy class of Z , factorising over g , i.e. $\varphi = f \circ g$ for some mapping $f : T \mapsto \mathbb{R}$. This mapping f is necessarily positive definite again, since g is positivity forcing. Because g is even *strongly* positivity forcing, the set $\mathcal{P}^{1,g}(\mathcal{Z})$ of positive definite functions on \mathcal{Z} factorising over g is a **Bauer simplex** whose extreme points are precisely the functions of the form $\rho \circ g$ with ρ being a bounded character on the semigroup $(T, +)$.

Definition 21. Let $\mathcal{P}^{c,g}(\mathcal{Z})$ denote the set of those functions in $\mathcal{P}^{1,g}(\mathcal{Z})$ which are continuous from below.

Lemma 22. $\mathcal{P}^{c,g}(\mathcal{Z})$ is an **extreme** subset of $\mathcal{P}^{1,g}(\mathcal{Z})$, i.e. there is no $\varphi \in \mathcal{P}^{c,g}(\mathcal{Z})$ which is equal to a non-trivial convex linear combination of functions in $\mathcal{P}^{1,g}(\mathcal{Z})$.

Proof. For decreasing functions, the property of being continuous from below is obviously an extreme one.

Lemma 23. $\mathcal{P}^{c,g}(\mathcal{Z})$ is a Bauer simplex with respect to the topology of full pointwise convergence, whose extreme points are precisely the below-continuous functions of the form $\rho \circ g$ with bounded characters ρ on T .

Proof. Immediate by the above and Lemma 22 .

In Lemma 23 , the below-continuous functions of the form $\rho \circ g$ with bounded characters ρ on T correspond to $\mu \in M_+^{1,e}(\mathcal{Y})$ with Q_{Z_1}, \dots, Q_{Z_m} μ -independent for any $Z_1, \dots, Z_m \in \mathcal{Z}$ with pairwise disjoint supports. The question is whether this extreme boundary is not only closed with respect to full pointwise convergence, but also with respect to weak convergence in $M_+^{1,e}(\mathcal{Y})$. So let μ_1, μ_2, \dots have the above property and $\mu_n \rightarrow \mu$. Then, as *pars pro toto*, $\mu_n(Q_{Z_1} \cap Q_{Z_2}) = \mu_n(Q_{Z_1}) \cdot \mu_n(Q_{Z_2})$, and in case that $\mu(Q_{Z_1} \cap Q_{Z_2}) = \mu(Q_{Z_1}) = \mu(Q_{Z_2}) = 0$ we get what we want. By the same countability and approximation argument as in $M_+^1(\mathbb{R}^d)$ this is enough to get the desired closedness property.

But for the uniqueness of mixtures of extreme points, there is still a problem with the two different topologies involved, which give rise to different sets of mixing Radon measures.

Acknowledgements

I am most grateful to my PhD supervisor Prof Dr Paul Ressel (Eichstätt) for having introduced me in a stimulating way into the powerful set of methods which has led to [HR1999] , [HR2000] , [H2003] , the present work and a DFG application on invariant probability measures on combinatorial structures.

I am also grateful to my present patron Prof Dr Joachim Gwinner (UniBw München) for the work environment and the freedom which allowed me to write [H2003] , the present paper and the mentioned DFG application.

3. References

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